AMS 131 25 Jun 18

1

this time: foundations

read: DeGroot & Schervish (4) 5 / ch. 1

next time:

ams131-summer18-01.courses.ucsc.edu

LSS tutor nkmills@ucsc.edu

webcast.ucsc.edu:

username: ams-131-1

password: probability-theory

all take-home

10 quizzes, 3 take-home tests

upload pdf to canvas.ucsc.edu

email subject line: "AMS 131 student"

official note-taker ...

office hours ...

timing of exams ...

probability = quantification of uncertainty

complete lack of information about something interesting to you

(1650) Pascal, Fermat | (classical approach) (1900)



9.45

population: all deer at ucsc today

sample: the observed deer

pop. size $N = 800$

disease? 1 = yes, 0 = no

$$\begin{bmatrix} 1_s \\ \& \\ 0_s \end{bmatrix} \xrightarrow{\text{at random}} \begin{bmatrix} 1_s \\ \& \\ 0_s \end{bmatrix} \quad n = 91$$

sample size

random variable

$Y_1$ = process of making a random draw ($n=1$) from pop

pop     sample

$$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \xrightarrow{\text{at random}} \begin{bmatrix} y_1 \end{bmatrix} \quad n = 1$$

outcome of draw

$$P_C ( Y_1 = 7 ) = ?$$

classical

elemental outcomes (EOs) $\{1, 2, 7\}$

judgment → equipossible

$$P_c(A) = \frac{\#\text{ of EOs favorable to } A}{\#\text{ of EOs}}$$ ③

$$P_c(\underline{Y}_1 = \overset{y_1}{7}) = \frac{1}{3}$$

① classical (1650)

$$P_c[\underline{Y}_1 \text{ is odd}) = \frac{2}{3}$$

③ frequentist (1840)

② Bayesian (1750)

context:

$\mathcal{B} = \{ B_1, B_2, \ldots, B_b \}$

(F, M both carriers)

(exactly 5 children)

P(1 or more Ts | $\mathcal{B}$) — conditional

babies

classical

probability: Equally Likely model (ELM)

T-S } ELM? yes

$$P_c(\text{normal}) = P_c(T\text{-}S) = \frac{1}{4}$$

$P(\text{carrier}) = \frac{1}{2}$

$P(1 \text{ or more T-S}) = ?$

---

$P(1 \text{ or more T-S}) = P(\text{exactly } 1 \; \boxed{\text{or}} \; \ldots \; \text{exactly } 5)$

---

$P(A \text{ or } B) \overset{?}{=} P(A) \; ? \; P(B)$

---

$P(1 \text{ or more}) = P[\boxed{\text{not}} \; (\text{exactly } 0)]$

---

$P(A) \overset{?}{=} P(\text{not } A)$

$P(\text{exactly } 0) =$

---

$P(\text{not T-S on 1st} \; \boxed{\text{and}} \; \cdot \; \boxed{\text{and}} \; \text{not T-S on 5th})$

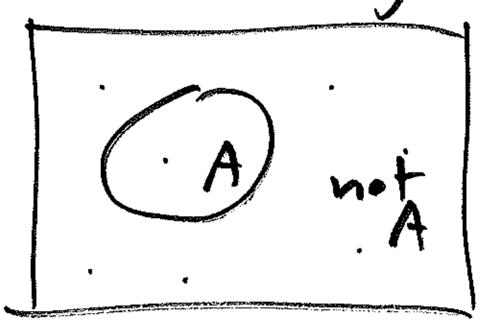
---

not

$0 \le P(A) \le 1$

(0%) (100%)

Venn diagram




$P(A) + P(\text{not } A) = 1$

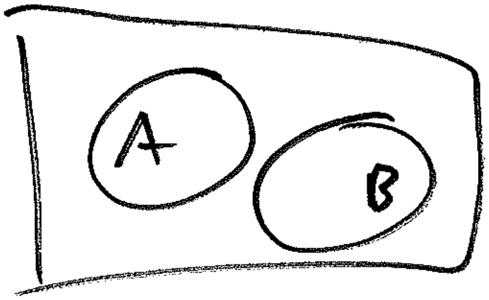
2 ways to compute P(A)

① direct: $P(A)$

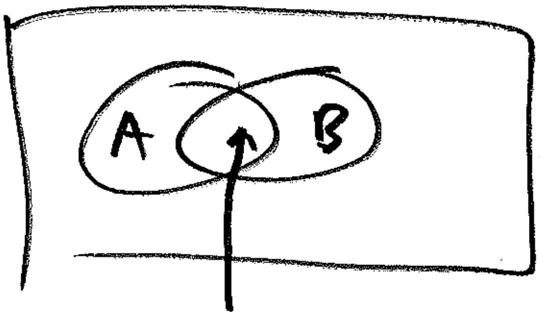
② indirect:

$$P(A) = 1 - \boxed{P(\text{not } A)}$$

or



$$P(A \text{ or } B) = P(A) + P(B)$$



A and B

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

"overlay"

$\left(\begin{array}{c} A, B \\ \text{no overlap} \end{array}\right) \longleftrightarrow \left(\begin{array}{c} A, B \text{ are} \\ \text{mutually} \\ \text{exclusive} \end{array}\right)$

and r.v.

$$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \xrightarrow[\text{random}]{\text{at}} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \quad n = 2$$

$$P(Y_1 = 7 \text{ and } Y_2 = 7) = ?$$

(at random with replacement) = (independent identically distributed (IID) sampling)

---

(at random without replacement) = (simple random sampling) (SRS)

---

IID vs. SRS

① SRS more informative than IID, but

② math harder with SRS

---

③ pop $N$ [ ] → sample $[\ ]_n$

if $n = 1$ SRS = IID

if $n \ll N$ SRS $\doteq$ IID

($\ll$ is a lot smaller than)

10.30

IID

pop

FID

$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ at random with rep'l. $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ $n = 2$

$P(Y_1 = 7 \boxed{\text{and}} Y_2 = 7) = ?$

ELM? yes

| $Y_1 \backslash Y_2$ | 1 | 2 | 7 |
|---|---|---|---|
| 1 | (1, 1) | (1, 2) | (1, 7) |
| 2 | (2, 1) | (2, 2) | (2, 7) |
| 7 | (7, 1) | (7, 2) | (7, 7) |

$P(Y_1 = 7 \text{ and } Y_2 = 7) = \frac{1}{9}$

$P(Y_1 = 7) = \frac{1}{3} = \frac{3}{9}$

$P(Y_2 = 7) = \frac{1}{3} = \frac{3}{9}$

$P(A \text{ and } B) \overset{IID}{=} P(A) \cdot P(B)$

$\frac{1}{9} = \frac{1}{3} \cdot \frac{1}{3}$

SRS

pop $\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix}$ $\xrightarrow{SRS}$ sample $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ $n = 2$

$P(Y_1 = 7 \text{ and } Y_2 = 7) = 0$

| | $Y_2$: 1 | 2 | 7 |
|---|---|---|---|
| $Y_1$: 1 | ~~(1,1)~~ | (1,2) | (1,7) |
| 2 | (2,1) | ~~(2,2)~~ | (2,7) |
| 7 | (7,1) | (7,2) | ~~(7,7)~~ |

$P(Y_1 = 7) = \frac{1}{3}$

$P(Y_2 = 7) = \frac{2}{6} = \frac{1}{3}$

previous theory doesn't work with SRS

$P(A \text{ and } B) = P(A) \cdot P(B)$

$P(Y_1 = 7 \text{ and } Y_2 = 7) = \cancel{\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}}$

$= 0$

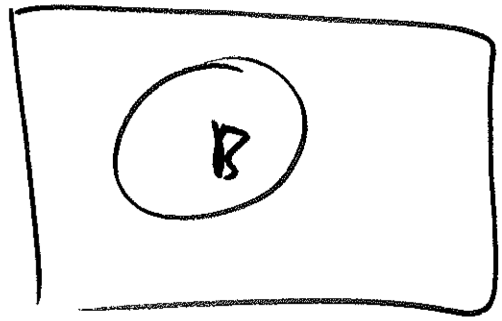
de Moivre (1718)

Bayes (1750)

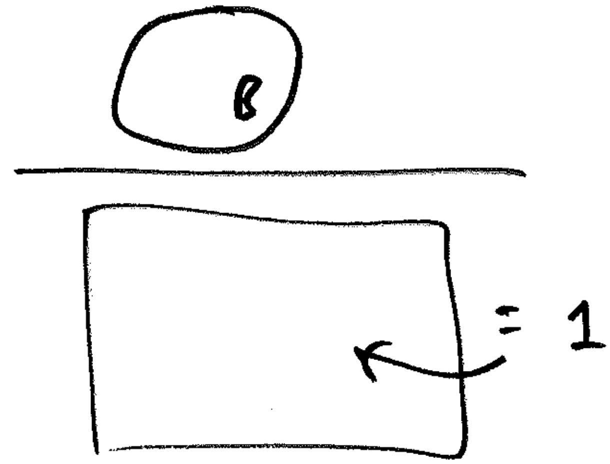
conditional probability

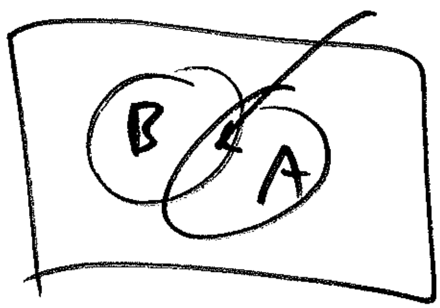
$P(B \text{ given } A) = ?$

$P(B \text{ given } A) = P(B \mid A)$

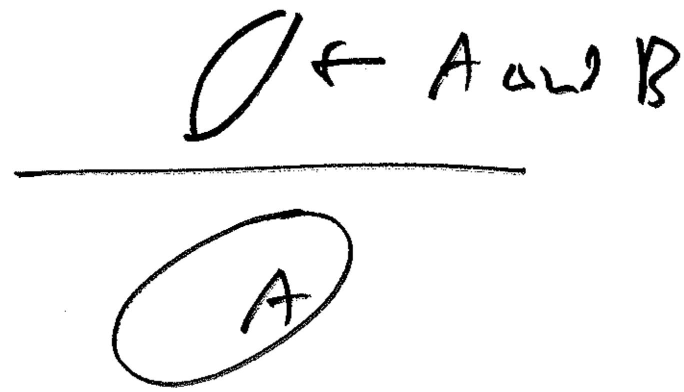



$P(B) =$




$P(B \mid A) =$




def.

$$P(B \mid A) = \begin{cases} \dfrac{P(A \text{ and } B)}{P(A)} & \text{if } P(A) \neq 0 \\ \text{undefined} & P(A) = 0 \end{cases}$$

(5a) $P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

$= P(B) \cdot P(A \mid B)$

$P(A \mid B) = \dfrac{P(A \text{ and } B)}{P(B)}$

(chain rule for prob.)

$P(A \text{ and } B \text{ and } C)$

$= P(A) \cdot P(B \mid A) \cdot P(C \mid A \text{ and } B)$

$P(\overbrace{X_1 = 7}^{A} \text{ and } \overbrace{X_2 = 7}^{B}) =$

$P(X_1 = 7) \cdot P(X_2 = 7 \mid X_1 = 7)$

$= \dfrac{1}{3} \cdot 0 = 0$ ✓

(IID) $P(X_1 = 7 \text{ and } X_2 = 7) =$

$P(X_1 = 7)\, P(X_2 = 7 \mid X_1 = 7)$

def. of indep (Bayesian)

A, B independent if information about A doesn't change probabilities about B & vice versa

under IID,

$P(\underline{\Sigma}_2 = 7 \mid \Sigma_1 = 7) = P(\Sigma_2 = 7)$

if A, B indep. $\begin{cases} P(B \mid A) = P(B) \\ P(A \mid B) = P(A) \end{cases}$

if A, B indep.

$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$

$= P(A) \cdot P(B)$ ✓

$= P(B) \cdot P(A \mid B) = P(A) \cdot P(B)$ ✓

def. A, B independent $\underset{\text{(iff)}}{\overset{\text{if and only if}}{\longleftrightarrow}}$ (12)

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

---

T-S solution

$P($1 or more T-S in 5 babies$)$

$= 1 - P($0 t-s babies$)$

| # T-S babies | if ELM applied |
|---|---|
| 0 | |
| 1 | |
| 2 | $P(1 \text{ or more})$ |
| 3 | |
| 4 | $= \frac{5}{6}$ |
| 5 | |

but ELM doesn't apply

$= 1 - P\left(\begin{matrix}\text{not TS}\\\text{on } 1^{st}\end{matrix} \text{ and } \begin{matrix}\text{not TS}\\\text{on } 2^{nd}\end{matrix} \text{ and} \ldots \text{and } \begin{matrix}\text{not TS}\\\text{on } 5^{th}\end{matrix}\right)$

IID

$= 1 - P\left(\begin{matrix}\text{not TS}\\\text{on } 1^{st}\end{matrix}\right) \cdot P\left(\begin{matrix}\text{not TS}\\\text{on } 2^{nd}\end{matrix}\right) \cdot \ldots \cdot P\left(\begin{matrix}\text{not TS}\\\text{on } 5^{th}\end{matrix}\right)$

IID

$= 1 - \left(1 - \frac{1}{4}\right)^5 \doteq 0.76$