Discussion Section 6

Probability Integral Transform

AMS 131 12 Jul 18

①

X continuous RV with CDF $F_X(x)$ and inverse CDF $F_X^{-1}(p)$ (quantile function)

$Y = F_X(X)$

$$F_Y(y) = P(Y \le y) = P[F_X(X) \le y]$$

can show that

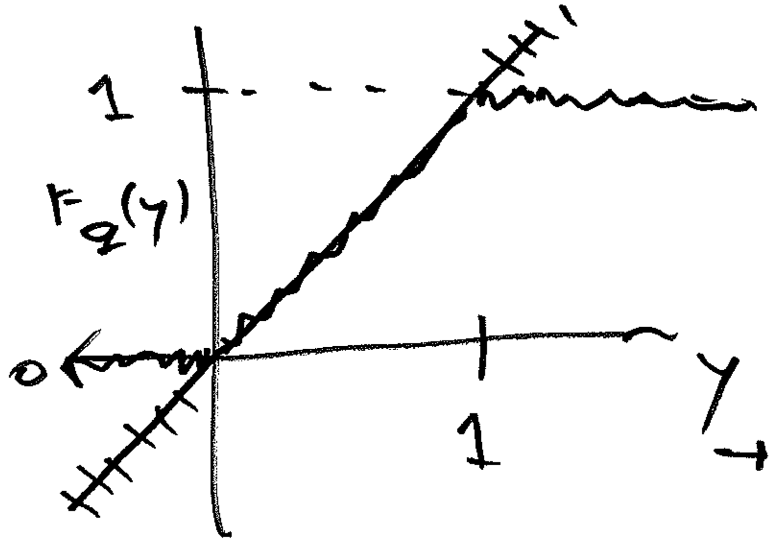
- F_X^{-1} is a strictly increasing function (i.e., if $p_1 < p_2$ then $F_X^{-1}(p_1) < F_X^{-1}(p_2)$)

so $$F_Y(y) = P\{F_X^{-1}[F_X(X)] \le F_X^{-1}(y)\}$$

$$= P[X \le F_X^{-1}(y)] = \underbrace{F_X[F_X^{-1}(y)]}_{= y}$$

$F_X(x) = P(X \le x)$

so $F_Y(y) = y$



$$F_Y(y) = \begin{cases} 0 & \text{for } y \le 0 \\ y & 0 \le y \le 1 \\ 1 & y \ge 1 \end{cases}$$

for $0 \le y \le 1$

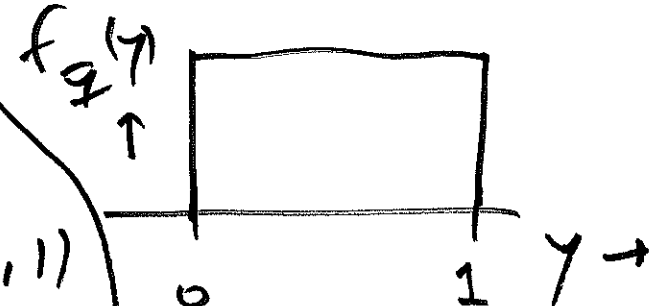
$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} y = 1$$

so PDF of Y is

$$f_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ 1 & 0 \le y \le 1 \\ 0 & y > 1 \end{cases}$$

if $Y = F_X(X)$

then $Y \sim \text{Uniform}(0,1)$ (!)



cont. X with CDF $F_X(x)$

$\boxed{Y = h(X)}$

$\rightarrow Y = X^2$

ex. $h(x) = x^2$

if $\underset{U}{Y} = F_X(X)$ then $Y \sim U(0,1)$

Uniform(0,1)

but it then follows that if

$U \sim \text{Uniform}(0,1) \to F_X^{-1}(U) = X$

$Y = F_X(X)$

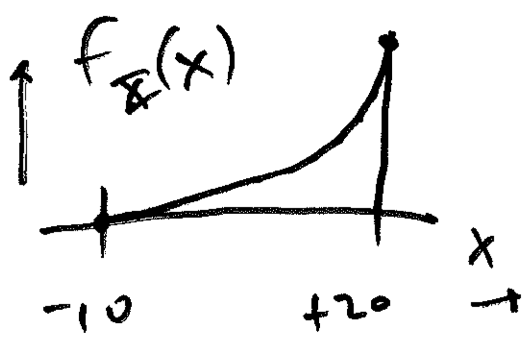
$F_X^{-1}(Y) = F_X^{-1}\left[F_X(X)\right] = X$

Prob. Int. Trans.

to generate a random draw from a RV X with CDF F_X, simply generate $U \sim \text{Uniform}(0,1)$ & calculate $F_X^{-1}(U)$

ex.

THT 1 problem 5(b)(ii)



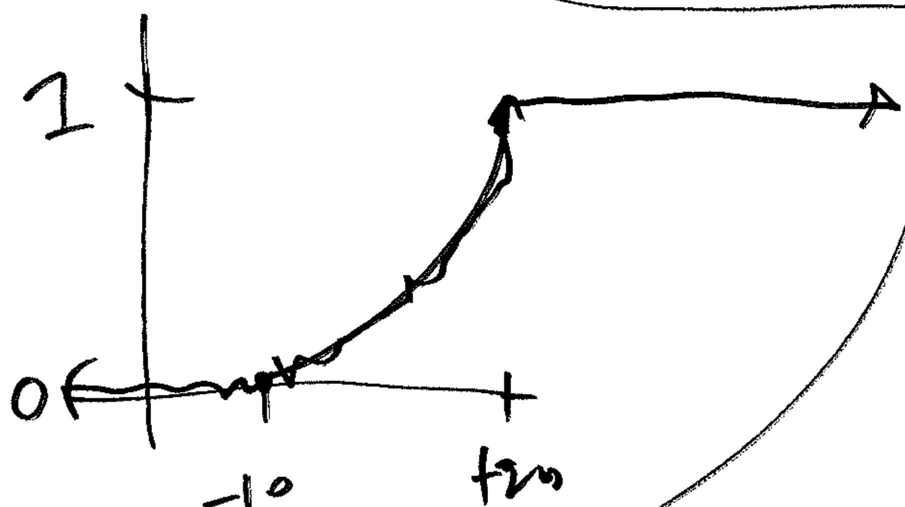
$$f_X(x) = \begin{cases} \frac{1}{9000}(x+10)^2 & \text{for } -10 \le x \le +20 \\ 0 & \text{else} \end{cases}$$

$$\int_{-10}^{+20} \frac{1}{9000}(x+10)^2 = \frac{1}{9000}\frac{(x+10)^3}{3}\Big|_{-10}^{+20}$$

for $-10 \le x \le +20$

$$F_X(x) = \int_{-10}^{x} \frac{1}{9000}(t+10)^2 \, dt = \frac{1}{27000}(x+10)^3$$

$$F_X(x) = \begin{cases} 0 & \text{for } x \le -10 \\ \dfrac{(x+10)^3}{27000} & -10 \le x \le +20 \\ 1 & x \ge +20 \end{cases}$$



$F_X^{-1}(p) = ?$

for

$-10 \le b \le +20$

$F_X^{-1}(p) = x \iff$

$$p = F_X(x) = \frac{(x+10)^3}{27000} \rightarrow 27000\,p = (x+10)^3$$

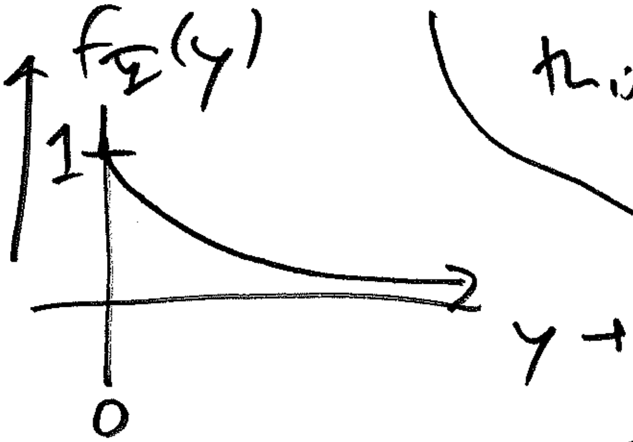
$$30\sqrt[3]{p} = x + 10 \rightarrow x = F_X^{-1}(p) = 30\,p^{1/3} - 10$$

(read DS pp. 169 - 172)

$\boxed{\text{DS p. 152 \#12}}$

PDF

$$f_Y(y) = \begin{cases} e^{-y} & y > 0 \\ 0 & \text{else} \end{cases}$$



this is the Exponential (λ) dist. with $\lambda = 1$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{(2y)^x e^{-2y}}{x!} & \text{for } x = 0, 1, \ldots \\ 0 & \text{else} \end{cases}$$

PMF

this is the Poisson (λ) dist with $\lambda = 2y$

(a) find $f_X(x)$:

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

def.
$$P(A|B) = \frac{P(A \text{ and } B)}{P(B)}$$

so $f_{X,Y}(x,y) = f_Y(y) \cdot f_{X|Y}(x|y)$

$$= \begin{cases} \dfrac{e^{-y}(2y)^x e^{-2y}}{x!} & \text{for } y>0 \text{ \& } x=0,1,\dots \\ 0 & \text{else} \end{cases}$$

---

so $f_X(x) = \displaystyle\int_0^\infty \frac{e^{-3y}(2y)^x}{x!}$ for $x = 0,1,\dots$

$\displaystyle\int_0^\infty y^k e^{-y}\,dy = k!$ for $k=0,1,\dots$

$= \dfrac{2^x}{x!}\displaystyle\int_0^\infty e^{-3y} y^x\,dy$

...

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \dots$$